Graph Theory (S.S)

History

It is no coincidence that graph theory has been independently discovered many times since it may quite properly be regarded as an area of applied mathematics. Indeed the earliest recorded mention of the subject occurs in the work of Euler, and although the original problem he was considering might be regarded as a somewhat frivolous puzzle, it did arise from the physical world.

Euler (1707-1782) became the father of graph theory as well as topology when in 1736 he settled a famous unsolved problem of the day called the Konigsberg Bridge problem.

There were two islands linked to each other and to the banks of the Pregel River by seven bridges as shown in the figure.

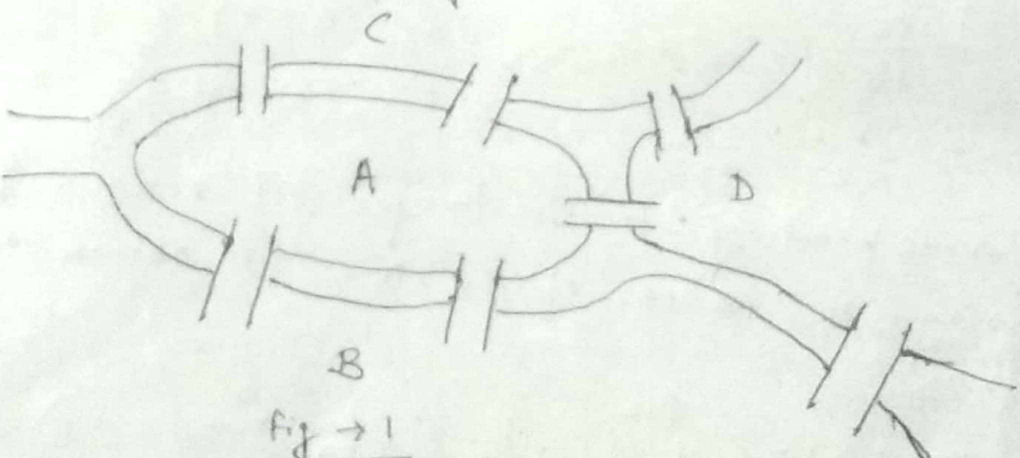

Fig → 1

The problem was to begin at any of the four land areas A, B, C, D, walk across each bridge exactly once and return to the starting point. One can easily try to solve this problem empirically, but all attempts must be unsuccessful.

In proving that the problem is unsolvable, Euler replaced each land area by a point and each bridge by a line joining the corresponding points, thereby producing a graph where the points are labelled to correspond to the four land areas.

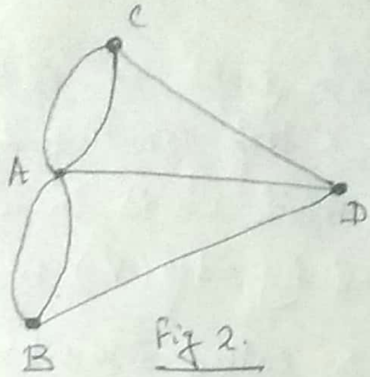

Fig 2.

Showing that the problem is unsolveable is equivalent to showing that the graph cannot be traversed in a certain way.

Euler generalized the problem and developed a critirion for a given ~~curve~~ graph to be so traversable namely that is connected and every point is incident with an even number of lines. While the above curve is connected but not every point is is incident with an even number of lines.

~~Let us~~

Graphs.

Let us start a formidable number of definitions in order to make available the basic concepts and terminology of graph theory.

A graph 'G' consists of two things 'Vertex' and 'edge', where the set of objects $V(G) = \{v_1, v_2 \cdots\}$ are called vertices points or nodes and the set of lines $E(G) = \{e_1, e_2 \cdots\}$ are called edges each of which joins either a pair of points or a simple point to itself.

Hence we denote a graph by $G(V, E)$ where V is the set of vertices and E is the set of edges of a directed graph.

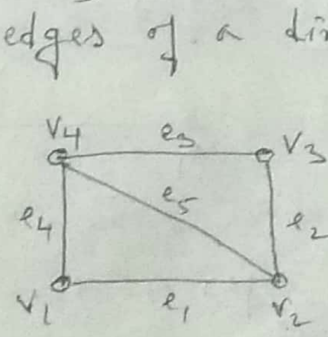

Fig 3.

Each edge 'e' is associated with two vertices say (v_i, v_j). The vertices v_i, v_j are called the end vertices or terminal vertices of edge e, where edge 'e' is incident at v_i as well as v_j.

Degree of a Vertex.

The degree of a vertex 'v' in a graph 'G' is the number of edges incident on 'v', with a loop counting 2 towards the degree of the vertex to which it is incident. The degree of v is denoted by deg(v) or d(v).

A vertex with degree 0 is called isolated.

In fig.1, the edges e_1 and e_2 are said to be adjacent since they have a common vertex (v_2).

Hence if two distinct edges are incident with a common vertex, then they are called adjacent.

The degree of a vertex 'v' in G is the number of edges incident at v.

Therefore in the fig.3, the degree of vertices are as follows $d(v_1) = 2$, $d(v_2) = 3$, $d(v_3) = 2$, $d(v_4) = 3$.

Simple Graph.

A graph that has neither self loops nor parallel edges is called a simple graph. Fig 3.

Parallel Edges.

In a graph, if a pair of vertices is connected by more than one edge, then those edges are called parallel edges.

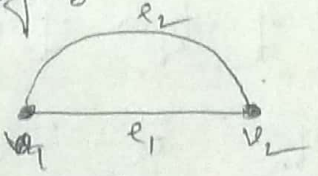

Fig 2.

In the above graph v_1 & v_2 are the two vertices connected by two edges e_1 & e_2. So e_1 & e_2 are parallel edges.

Multiple Graphs.

The graphs in which the same two vertices are joined by more than one edge, are called Multigraphs.

In the adjacent figure v_1 and v_2 are joined by parallel edges e_1 & e_2. Further an edge e_6 joined a vertex to itself is called a loop. Therefore e_6 contributes two degrees of v_4. Therefor the degrees of the Multigraph are as follows

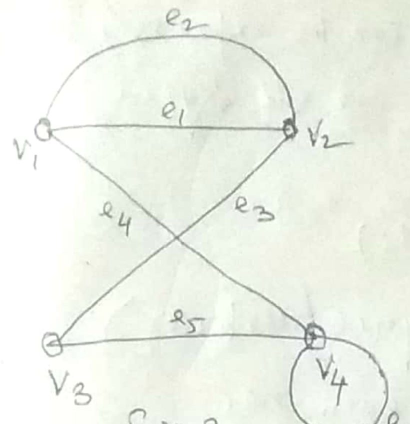

Fig 3.

$d(v_1) = 3$, $d(v_2) = 3$, $d(v_3) = 2$; $d(v_4) = 4$.

A vertex with degree one is called pendant.

v_5 is pendant vertex and v_6 is isolated vertex in Fig. 4.

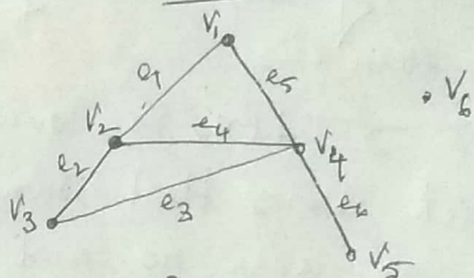

Fig 4.

Weighted graph.

The graph G is called a weighted graph if each edge e of G is assigned a non-negative number $w(e)$ called the weight or length of v

Finite and infinite Graph.

A multigraph is said to be finite if it has a finite number of vertices and a finite no. of edges. Otherwise it is an infinite graph.

The finite graph with one vertex and no edges i.e, a single point, is called a trivial graph.

Handshaking Lemma (Theorem).

Let $G = (V, E)$ be any graph with e edges and n number of vertices, then $\sum_{i=1}^{n} d(v_i) = 2e$.

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Now the total degree of all the vertices can be expressed as the sum of degrees of even degree and odd degree vertices i.e,

$$\sum d(v_j) + \sum d(v_k) = \sum_{i=1}^{n} d(v_i)$$

i.e, Σ(even degree vertices) + Σ(odd degree vertices)

$$\Rightarrow \sum d(v_j) + \sum d(v_k) = 2e = \text{even no.}$$

But the sum of even degree vertices is always even i.e, $\sum d(v_j)$ is always even and $2e$ is also even resulting the sum of odd degree vertices should be even.

$$\Rightarrow \sum d(v_k) = \text{even no.}$$

which shows that sum of the degrees of odd vertices is an even no. and which is possible only when the number of odd degree vertices is even.

Corollary.

In any graph, the number of vertices of odd degree is even.

Prob. 1.

How many nodes are necessary to construct a graph with exactly 6 edges in which each node is of degree 2.

Solⁿ: Given, graph with exactly 6 edges and each node is of degree 2.

$\therefore$ $2 \times$(No. of edges) = (Number of nodes) $\times 2$.

$\therefore$ $2 \times 6 = 2$ Number of nodes.

$\therefore$ Number of nodes $= 6$.

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